On the Distribution of Parity in the Partition Function

By Thomas R. Parkin and Daniel Shanks

1. Introduction. Let $p(n)$ be the number of (unrestricted) partitions of n, and define $p(0) = 1$. Then $p(n)$ is generated by

$$(1) \quad \sum_{n=0}^{\infty} p(n)x^n = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)} = \left\{ 1 + \sum_{n=1}^{\infty} (-1)^n [x^{n(3n-1)/2} + x^{n(3n+1)/2}] \right\}^{-1} .$$

There is little known about $p(n)$ modulo 2; in particular, there are no known criteria for the parity of $p(n)$ comparable in simplicity with Ramanujan's famous *sufficient* condition for divisibility by 5:

$$(2) \qquad 5 \mid p(5k + 4) .$$

Kolberg [1] proved, but by contradiction and without identifying the arguments n, that infinitely many $p(n)$ are even, and infinitely many are odd. His proof is almost as simple as Euclid's proof that there are infinitely many primes, but like that proof it offers only very little more in the way of exact information concerning questions of distribution.

From Gupta's tables [2], [3] we find the following cumulative distribution into odds and evens for $0 \leqq n \leqq 499$.

	$n \leqq 99$	$n \leqq 199$	$n \leqq 299$	$n \leqq 399$	$n \leqq 499$
Odds	58	111	171	222	277
Evens	42	89	129	178	223

In the absence of any known reason to the contrary, and because of the rather unsmooth recursion for $p(n)$ implied by (1), it would be natural to guess that the evens and odds are equinumerous, i.e., that the ratio of their counts has the limit 1 as the upper bound for $n \to \infty$. But the early preponderance of the odds, as just tabulated, would make us hesitate to conjecture that this is true. Nonetheless, it seemed to us not unlikely that this early preponderance might wash out as later returns came in (from upstate, so to speak). But it does seem unlikely that a theoretical proof of this could be attained with known techniques.

We have therefore examined the question empirically with a computer, and have put an even stronger question. Consider the number $m = 1.74264258\cdots$, which when written in binary:

$$(3) \qquad m = 1.10111110000111011101 \cdots ,$$

has its kth bit to the right of the binary point 0 or 1 according as $p(k)$ is even or odd. (m stands for Major MacMahon.) We now ask if m is normal with respect to the base 2. If so, this not only implies the previously supposed equinumerosity, but

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also implies that all possible pairs, 00, 01, 10, and 11, have an asymptotic density of $\frac{1}{4}$, etc.

Here, however, we must note that the corresponding proposition modulo 5 is definitely false. Thus, if

$$r = 1.12302102021210112002 \cdots \tag{4}$$

is a number written in quinary with its kth place $\equiv p(k)$ modulo 5, we know from (2) that r is *not* normal. In fact, r is not even *simply* normal since it is further known that more than 20% of the $p(n)$ are divisible by 5. For, in addition to (2), Morris Newman shows, in the following paper [4], that

$$5|p(5\cdot 19^4k + 15147) \tag{5}$$

also, and still other independent linear functions also have this property. (A. O. L. Atkin has obtained more general results; these will appear in [5].)

One of our reasons for stressing this failure modulo 5 is because of the character of our main problem. Suppose, for instance, that our empirical investigation shows that parity does appear to be equinumerous, and even normal. Then one might well remark: "So what? Isn't that what one expects?" But the failure modulo 5 puts the problem in a more interesting light.

We have determined the parity of $p(n)$ up to $n = 2,039,999$. In what follows we will indicate our method, our results, and some related investigations.

2. Notation and Nomenclature. Let a_n be the nth bit of m in (3):

$$a_n \equiv p(n) \qquad (\text{mod } 2)\ . \tag{6}$$

Let the finite sequence

$$a_m a_{m+1} a_{m+2} \cdots a_{m+k-1}$$

be called the mth k-tuple. Thus 1101 is the 0th 4-tuple and 11111 is the 3rd 5-tuple. There are 2^k possible types of k-tuple, and let us designate these 2^k types by the integer, which, when written in binary, is the k-tuple itself. Thus 1101 is the 13th type of 4-tuple and 11111 is the 31st type of 5-tuple. Let

$$\sum_t^{(k)} (n)$$

be the number of t type k-tuples that appear to the left of, but *not including*, the nth k-tuple. (We find it convenient, because of (11) below, to count the 0th k-tuple here, and therefore to omit the nth, so that the argument n in $\sum_t^{(k)} (n)$ means that n k-tuples have been counted.) Thus, from (3),

$$\sum_0^{(2)} (10) = 2\,, \qquad \sum_1^{(2)} (10) = 1\,, \qquad \sum_2^{(2)} (10) = 2 \qquad \sum_3^{(2)} (10) = 5$$

and referring to our previous table,

$$\sum_0^{(1)} (500) = 223\,, \qquad \sum_1^{(1)} (500) = 277\ .$$

Then equinumerosity means

(7) $$\sum_0^{(1)} (n) \sim \sum_1^{(1)} (n) \sim \frac{1}{2} n,$$

while the stronger normality means that

(8) $$\sum_t^{(k)} (n) \sim 2^{-k} n$$

as $n \to \infty$ for all t and all k.

Note that if one has counted the k-tuples $\sum_t^{(k)} (n)$, one can obtain the counts of j-tuples with $j < k$ simply by addition. Thus

$$\sum_0^{(8)} (n) + \sum_1^{(8)} (n) = \sum_0^{(7)} (n),$$

and generally

(9) $$\sum_{2t}^{(k)} (n) + \sum_{2t+1}^{(k)} (n) = \sum_t^{(k-1)} (n)$$

for all k and all t.

To test normality we have counted the 256 types of 8-tuples out to $n = 2 \cdot 10^6$, and we deduced from these the counts, successively, of 7-tuples, 6-tuples, etc.

3. Computing the Parity Individually or En Masse. That the first two terms of equation (1) are equal is fairly obvious. For the simplest proof of the equality of the second and third terms, see [6]. Together, these equations imply Euler's recurrence: For $n \geqq 1$,

(10) $$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots + (-1)^{i+1} p(n - e_i)$$

where $e_i = \frac{1}{2} i(3i \mp 1)$, and where the series breaks off just before $n - e_i$ becomes negative. One may thus compute the a_n *en masse* by recurrence using (10) modulo 2. For n large about $\frac{2}{3} (6n)^{1/2}$ terms are needed to compute a_n if the previous a_{n-e_i} are already known.

But MacMahon [7] found the more efficient recurrences:

(11) $$\begin{aligned} a_{4n} &\equiv a_n + a_{n-7} + a_{n-9} + \cdots + a_{n-\alpha_i} \quad \text{with} \quad \alpha_i = i(8i \mp 1) \\ a_{4n+1} &\equiv a_n + a_{n-5} + a_{n-11} + \cdots + a_{n-\beta_i} \quad \text{with} \quad \beta_i = i(8i \mp 3) \\ a_{4n+3} &\equiv a_n + a_{n-3} + a_{n-13} + \cdots + a_{n-\gamma_i} \quad \text{with} \quad \gamma_i = i(8i \mp 5) \\ a_{4n+6} &\equiv a_n + a_{n-1} + a_{n-15} + \cdots + a_{n-\delta_i} \quad \text{with} \quad \delta_i = i(8i \mp 7) \end{aligned} \pmod 2.$$

(Note that $4n + 2 = 4(n - 1) + 6$, but the formulas are neater as given.) We will give a proof of (11) presently. For now, let us note the savings possible.

(1) The number of terms for a_n (not a_{4n}) with n large is now $\sim\frac{1}{4} (2n)^{1/2}$ so that the use of (11) requires only $\sqrt{3/8} = 0.2165$ as much arithmetic as the use of (10).

(2) To compute a_n out to $n = N$ we now need to save the a_n only to $n = [N/4]$, so that only 0.25 as much storage is necessary.

Aside from this more efficient computation *en masse*, there also arises the possibility of iterating (11), and thus of computing an individual a_n with no mass storage whatsoever, since each application of (11) reduces the arguments by a factor of 4. We will discuss this possibility briefly later.

4. MacMahon's Congruences. In [7] MacMahon gave a proof of (11) based upon self-conjugate partitions, and in [8] he used (11) to compute the parities out to $n = 1000$. Subsequently, independently, and in effect, but not explicitly, G. N. Watson [9] reproved (11) using theta functions. Still later, H. Gupta [10] gave still another proof, this time using Ramanujan's tau function.

Perhaps the most direct proof, since it involves knowledge of none of these special concepts or functions, is this: Since

$$\frac{1}{1-x^n} = 1 + x^n + x^{2n} + \cdots \equiv 1 - x^n + x^{2n} - \cdots = \frac{1}{1+x^n} \pmod 2$$

we have

$$\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots} \equiv \frac{1}{(1-x)(1+x^2)(1-x^3)(1+x^4)\cdots}$$
$$= \frac{(1-x^2) \qquad (1-x^4)\cdots}{(1-x)(1-x^4)(1-x^3)(1-x^8)\cdots} \pmod 2$$

Thus

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} \equiv \prod_{n=1}^{\infty} \frac{1}{1-x^{4n}} \cdot \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^{2n-1}} \pmod 2 .$$

Since the product on the right equals $\sum_{n=0}^{\infty} x^{n(n+1)/2}$ (see [11] for the shortest proof) we have

$$(12) \qquad \sum_{n=0}^{\infty} p(n)x^n \equiv \sum_{n=0}^{\infty} p(n)x^{4n} \sum_{n=0}^{\infty} x^{n(n+1)/2} \pmod 2 ,$$

and comparing like powers of x, congruences (11) follow quickly.

It may be of interest to indicate the quite extraneous considerations that led us to this problem. One of us was in the process of reviewing [12] *The Groups of Order* 2^n ($n \leqq 6$), by Marshall Hall, Jr. and James K. Senior, Macmillan, New York, 1964. The abelian groups there are designated as belonging to a family Γ_1, and the number of such groups of order 2^n is, of course, $p(n)$. It may be noted, see pages 103–104, that the lattice diagrams of these groups suggest that they fall into dual pairs. The question of whether $p(n)$ is even or odd is therefore the question of whether there are an even or odd number of lattices which are self-dual.

This leads one to consider self-conjugate partitions and thus to rediscover (11) with (essentially) MacMahon's proof. But the proof above is somewhat simpler. Naturally, after having "discovered" the efficient congruences (11), one is eager to exploit them.

5. Normality. We show in Table 1 the value of $\frac{1}{2}m = .676\cdots$ in octal to 3200 places. In this one can read a_n for $0 \leqq n \leqq 9599$. We have placed in the UMT file of this journal the complete 213-page value of $\frac{1}{2}m$ out to $n = 2{,}039{,}999$. In Tables 2 and 4 we list the counts of the 8-tuples $\sum_t^{(8)}(n)$ for $t = 0(1)255$ and $n = 10^6$ and $2\cdot 10^6$, respectively. For example,

TABLE 1

$\frac{1}{2}\, m$ *in octal*

67607351411575306750	64356307277504550463	24205071725165566306	35567627437357662367
52075125115565372151	32556214707510262064	74742346215355630006	65547353102627735265
52472572460244444074	54600470361750770362	24137136754342411332	63706045316542106255
57574736154564606003	04514216671271263464	30437317535150733746	34477640506526360474
70234124026472604422	46057112300665735235	23746771227074257033	17045235273122137234
76124515670421254470	51014111160740535051	04236772036575037605	23504515104712421711
27651203441723714422	63323540244057731516	11415364502417657566	33254102750012352754
04434110474336540616	15322441772400361321	52303022337744760572	76104532306310011704
51142356224063407521	44773144531561231570	76753424122165147601	62240477146062772347
16005531126570104227	27735320626572113112	45036323365111753212	13223433776134303032
05025345333341504605	63656175572500747517	47514542420576275600	01700121000107777762
15331236127410160740	07077301161170211467	74545503647320367205	54303012012775520632
56144142715775447715	27172337665601324136	07600107350045434371	24455517156073252460
53115045524311367347	56243760213136163715	53245351011021132166	35622426073724542207
64510230467314474640	00400546553543773652	54616526743675576717	55612031220513675752
43712327727127755175	57753242235345135240	32460172054673131773	54336616066104337675
46101150772251377475	77347104211017527574	33233224161166621156	70523106406340672050
06150773562137162006	25007653541211041752	62064642226575020402	71620207046201571750
01150556064430367736	64217440227461647132	14321635133474117034	17561673367621642672
41014440104336463722	40564752110450562544	37342142326264055550	07310155647232147267
53363677515613644721	72351304560311623032	37746104664565550655	75305773746442177424
02046361016101724144	53700262570014103235	26402666744147506237	02002006635661130552
15360635325746445174	47632717333762167070	60564537447570234572	14604271330106566115
63210417513214452117	02035204421122043155	07331563030216166115	01332135354234530332
12751052105301330543	27140611372466221323	04305754214361632623	60365302353063037704
54427355237140710436	62176465233001531343	54403600354446765574	52265612664113100251
27154164545277474753	10614163424162065311	74736317342364320120	65422664664421151100
54050244651056150674	06236514734075550754	04740651072201230125	05504240556024413364
01136400452407060227	56172660743015522007	47446066723226512062	53462436567320613063
50523762360563760151	26540472224004007547	33204700566054362315	63557313704025373351
23661011544736731411	04175704734021430707	50527605112461175050	35461476327527274757
44747542056131426250	54514432034322216367	25415107215526033072	55525420702776243171
41156036576000773050	05046557414447646525	46375255271600677422	02774446167365772372
72770177114464213713	04401641714016215557	76420275053203364566	73236306733061050726
15524633035661166233	32772027322215440044	35223325453545374075	05643420561424440501
02545457500673746470	60471766306630510526	25721603267153212213	53026413576307750631
02335351707665521260	26426474561002173553	61134266254706244775	47227205027224020467
05731321753571463002	45731607052266426325	57567676137434744262	63006105570212531053
07630036513537322375	23147651222100123115	30314754441254376363	73317652177733501046
60731215077136554057	64003311310360420224	76047722131057601435	70543425302260175774

TABLE 2

OCTUPLE COUNTS 0 TO 999999

0	1	2	3	4	5	6	7
3952	3948	3867	3977	3949	3940	3981	3978
3892	3934	3884	3894	3977	3980	3910	3925
3891	3945	3911	3965	3881	3939	3803	4014
3991	3914	3899	3972	3977	3903	3856	3910
3981	3790	3873	3926	3859	3877	3827	4015
3958	3726	3950	3900	3918	3801	3918	3920
3939	3910	3892	3840	3850	3847	3830	4047
3982	3965	3943	3924	3915	3905	3777	3930
3908	4019	3861	3899	3909	3953	3978	3932
3865	3887	3823	3841	3901	3867	4082	3899
3962	3957	3774	3909	3956	4038	3878	3974
3948	3837	3883	3839	3875	4048	3800	3914
3981	3935	3878	3918	3840	3885	3778	3833
3886	3894	3887	3897	3786	3863	3939	3929
3932	3935	3924	3906	4091	3941	3951	3871
3852	3828	3928	3909	3822	3899	3957	3861
3948	3896	4022	3982	3877	3838	3976	3857
3944	3942	3936	3923	3928	3891	3970	3841
3880	3854	3826	3877	3803	3911	3916	3824
3858	3818	3798	3905	3970	3964	3964	3797
3946	3970	3989	3984	3893	3787	3941	3966
3961	3957	4044	3952	3867	3921	4005	3794
3977	3886	3833	3771	3930	3937	3819	3821
3885	3865	4089	3898	3765	3932	3944	3888
3936	3985	3854	3934	3977	3906	3841	3879
3869	3816	3891	3899	3775	3836	3852	3862
3954	4016	3906	3998	3962	3958	3910	3825
3915	3767	3984	3801	3875	3939	3897	3918
3940	3853	4005	3802	3845	3905	3833	3881
4084	4010	4033	3838	3896	3922	3875	3885
3861	3872	3826	3808	4003	3930	3867	3889
3881	3806	4005	3847	3865	3953	3861	3983

TABLE 3

K-TUPLE COUNTS 0 TO 999999

7900	7844	7889	7959	7826	7778	7957	7835
7836	7876	7820	7817	7905	7871	7880	7766
7771	7799	7736	7842	7684	7850	7719	7838
7849	7732	7697	7877	7947	7867	7820	7707
7927	7760	7862	7910	7752	7664	7768	7981
7919	7683	7994	7852	7785	7722	7923	7714
7916	7796	7725	7611	7780	7784	7649	7868
7867	7830	8032	7822	7680	7837	7721	7818
7844	8004	7715	7833	7886	7859	7819	7811
7734	7703	7714	7740	7676	7703	7934	7761
7916	7973	7680	7907	7918	7996	7788	7799
7863	7604	7867	7640	7750	7987	7697	7832
7921	7788	7883	7720	7685	7790	7611	7714
7970	7904	7920	7735	7682	7785	7814	7815
7793	7807	7750	7714	8094	7871	7818	7760
7733	7634	7933	7756	7687	7852	7818	7844
15744	15848	15604	15792	15712	15637	15776	15646
15570	15578	15534	15557	15581	15574	15814	15527
15687	15772	15416	15749	15602	15846	15507	15637
15712	15336	15564	15517	15697	15854	15517	15539
15848	15548	15745	15630	15437	15454	15379	15695
15889	15587	15914	15587	15467	15507	15737	15529
15709	15603	15475	15325	15874	15655	15467	15629
15600	15464	15965	15578	15367	15689	15539	15662
31592	31396	31349	31422	31148	31091	31155	31341
31459	31165	31448	31144	31048	31081	31551	31056
31396	31375	30891	31074	31476	31501	30974	31266
31312	30809	31529	31096	31064	31543	31056	31201
62988	62771	62239	62496	62624	62592	62129	62607
62771	61965	62977	62240	62112	62625	62607	62257
125759	124735	125216	124736	124736	125217	124737	124864
250494	249952	249953	249601	500446	499554		

TABLE 4

OCTUPLE COUNTS 0 TO 1999999

0	1	2	3	4	5	6	7
7841	7703	7689	7869	7851	7791	7844	8016
7706	7936	7745	7830	7916	7726	7878	7902
7747	7817	7868	7964	7784	7815	7748	7912
7931	7855	7749	7778	7947	7845	7778	7834
7871	7641	7747	7839	7823	7874	7804	7948
7891	7623	7839	7745	7884	7694	7845	7849
7938	7888	7791	7744	7653	7838	7728	7793
7938	7929	7893	7846	7820	7881	7631	7810
7720	7950	7738	7776	7838	7862	7938	7779
7783	7804	7663	7821	7835	7795	8036	7794
7844	7931	7663	7827	7948	7924	7612	7888
7932	7680	7791	7703	7721	8001	7746	7836
7989	7865	7834	7837	7786	7813	7609	7726
7691	7725	7835	7835	7641	7726	7763	7821
7873	7865	7798	7815	7993	7890	7823	7787
7745	7736	7792	7816	7751	7829	7887	7725
7703	7855	7953	7991	7791	7734	7798	7764
7858	7896	7854	7830	7870	7801	7914	7710
7765	7769	7829	7788	7730	7769	7830	7782
7895	7680	7742	7743	7920	7894	7923	7607
7799	7873	7953	7878	7764	7610	7826	7882
7884	7867	8033	7755	7728	7800	7877	7733
7916	7783	7808	7591	7763	7832	7639	7791
7800	7684	7990	7764	7711	7727	7949	7802
7838	7994	7837	7786	7916	7822	7733	7845
7751	7813	7836	7791	7740	7690	7778	7736
7828	7900	7711	7881	7803	7864	7916	7722
7767	7719	7804	7727	7763	7753	7692	7915
7843	7758	7904	7741	7778	7814	7821	7788
8037	7867	7832	7804	7845	7805	7753	7785
7728	7780	7794	7794	7911	7746	7827	7751
7763	7802	7865	7762	7814	7798	7725	7941

TABLE 5

K-TUPLE COUNTS 0 TO 1999999

15544	15558	15642	15860	15642	15575	15642	15780
15564	15832	15599	15660	15786	15527	15792	15612
15512	15586	15697	15752	15514	15584	15578	15694
15826	15535	15491	15521	15867	15739	15701	15441
15670	15514	15700	15717	15587	15484	15630	15830
15775	15490	15872	15500	15612	15494	15722	15582
15854	15671	15599	15335	15416	15670	15367	15584
15738	15613	15883	15610	15531	15608	15580	15612
15553	15944	15575	15562	15754	15684	15671	15624
15534	15617	15499	15612	15575	15485	15814	15530
15672	15831	15374	15708	15751	15788	15528	15610
15699	15399	15595	15430	15484	15754	15438	15751
15832	15623	15738	15578	15564	15627	15430	15514
15728	15592	15667	15638	15486	15531	15516	15607
15601	15645	15592	15609	15904	15636	15650	15538
15508	15588	15657	15578	15565	15627	15612	15666
31102	31502	31217	31422	31396	31259	31313	31404
31098	31449	31098	31272	31361	31012	31606	31142
31184	31417	31071	31460	31265	31372	31106	31304
31525	30934	31086	30951	31351	31493	31139	31192
31502	31137	31438	31295	31151	31111	31060	31344
31503	31082	31539	31138	31098	31025	31238	31189
31455	31316	31191	30944	31320	31305	31017	31123
31246	31201	31540	31188	31096	31235	31192	31278
62604	62639	62655	62717	62547	62370	62373	62748
62601	62531	62637	62410	62459	62037	62844	62331
62639	62733	62262	62404	62585	62677	62123	62427
62771	62135	62625	62140	62447	62728	62331	62470
125243	125372	124917	125121	125132	125047	124496	125175
125372	124666	125262	124550	124906	124765	125175	124801
250615	250038	250179	249671	250038	249812	249671	249976
500653	499850	499850	499647	1000503	999497		

$$\sum_{0}^{(8)} (10^6) = 3952 \quad \text{and} \quad \sum_{12}^{(8)} (2\cdot 10^6) = 7916\,.$$

These tables are read first across, and then down, for increasing t.

From Tables 2 and 4 we compute the counts of k-tuples for $k = 7, 6, \cdots, 1$ at $n = 10^6$ and $2\cdot 10^6$, respectively. This is done by use of the recursion (9), and the results are listed in Tables 3 and 5 in the obvious way. Thus

$$\sum_{0}^{(7)} (10^6) = 7900\,, \qquad \sum_{1}^{(6)} (10^6) = 15848\,, \qquad \sum_{2}^{(5)} (2\cdot 10^6) = 62655\,.$$

The initial impression of this data is that no type of k-tuple is favored over other types, that the various types are equidistributed, and that the data here is consistent with the hypothesis of normality. We have attempted no elaborate statistical tests of this, but we did examine Good's *psi-square* serial test [13], [14] to a limited extent. Let

$$(13) \qquad \psi_k^{\,2} = 2^k n^{-1} \sum_{t=0}^{2^k-1} \left(\sum_{t}^{(k)} (n) - 2^{-k} n \right)^2 .$$

Good showed that if the bits of a binary number are *random*, then ψ_k^2 has an *expectation* $2^k - 1$. We list these ψ_k^2 for $k = 1(1)6$ and $n = 10^6, 2\cdot 10^6$ together with their expectation in Table 6.

TABLE 6

k	$n = 10^6$	$n = 2\cdot 10^6$	*Expect.*
1	0.796	0.506	1
2	1.631	1.192	3
3	7.737	2.662	7
4	23.106	9.429	15
5	44.329	21.770	31
6	87.733	56.850	63

Now note: We are testing here for *randomness*, but we are really interested in *normality*. The former implies the latter, but what of the converse? The data in Table 6 is consistent with randomness, and therefore also with normality. At $n = 2\cdot 10^6$ (but not at $n = 10^6$) the distribution is even "too good." It seems to us conceivable (but admittedly, we are now going somewhat beyond our competence) that real numbers may exist with the ψ_k^2 *consistently* too small. While such behavior would *not* be *random*, it could still imply normality—in fact, the smaller the ψ_k^2 are, the better.

6. Equinumerous Evens and Odds. Turning now to $k = 1$ in greater detail—and the question whether even and odd partition numbers are equinumerous—we list in Table 7 the number of *odds*, $\sum_1^{(1)} (n)$, and the *ratio* of odds to evens $\sum_1^{(1)} (n)/\sum_0^{(1)} (n)$ for $n = 50{,}000(50{,}000)2\cdot 10^6$.

Since these steps $\Delta n = 50{,}000$ are large and therefore do not allow a completely accurate picture of the variations in the *ratio* function, we supplement Table 7 with the description in Table 8. This lists 11 regions, A through K, within each of which the *ratio* remains continually greater than 1, or continually less than 1. Thus, the early preponderance of the odds, that we already noted, continues throughout region A until $n = 6672$. Between these regions there are many small oscillations of the ratio function around the value 1. For example, between regions G and H, the difference:

$$\text{odds} - \text{evens}$$

varies between $+56$ and -65, and the ratio equals 1 for 176 different values of n (including, as in Table 7, $n = 400{,}000$).

TABLE 7

$n \cdot 10^{-4}$	*Odds*	*Ratio*	$n \cdot 10^{-4}$	*Odds*	*Ratio*
5	25016	1.00128	105	524597	0.99847
10	50200	1.00803	110	549632	0.99866
15	75041	1.00109	115	574646	0.99877
20	99766	0.99533	120	599770	0.99923
25	124703	0.99526	125	624669	0.99894
30	149758	0.99678	130	649700	0.99908
35	175105	1.00120	135	674581	0.99876
40	200000	1.00000	140	699672	0.99906
45	225123	1.00109	145	724763	0.99935
50	250016	1.00012	150	749745	0.99932
55	274917	0.99940	155	774859	0.99964
60	299972	0.99981	160	799757	0.99939
65	324951	0.99970	165	824694	0.99926
70	349834	0.99905	170	849627	0.99912
75	374718	0.99850	175	874724	0.99937
80	399531	0.99766	180	899622	0.99916
85	424656	0.99838	185	924804	0.99958
90	449744	0.99886	190	949733	0.99944
95	474475	0.99779	195	974570	0.99911
100	499554	0.99822	200	999497	0.99899

TABLE 8

Region	*Limits*	*Ratio*	*Extreme* $\psi_1(n)$	*At n*
A	1–6671	>1	$+1.996$*	1230*
B	16287–48781	<1	-1.662	21017
C	49185–151211	>1	$+2.882$	78823
D	162951–332867	<1	-1.684	241706
E	333373–363347	>1	$+0.553$	347684
F	363769–375013	<1	-0.158	367246
G	376961–395293	>1	$+0.204$	386259
H	406565–494241	>1	$+0.692$	434150
I	538051–601509	<1	-0.499	569769
J	637169–645423	>1	$+0.154$	641119
K	646475–2040000+	<1	-1.165	812968

* Only $n > 1000$ examined here.

Consistent with the definition (13) is the designation $\psi_1(n)$ for the *normalized difference*:

$$\text{(14)} \qquad \frac{\text{odds} - \text{evens}}{\sqrt{n}} = \frac{\sum_1^{(1)}(n) - \sum_0^{(1)}(n)}{\sqrt{n}} = \psi_1(n)\,.$$

As in the previous section, our main interest here is not so much in the distribution of $\psi_1(n)$ as in its extreme values, and in Table 8 we list the extreme value it takes on in each interval. For instance, in region B, at $n = 21017$ there are 10629 evens and 10388 odds for an extreme value

$$\psi_1(21017) = -1.662\,.$$

In regions E through J parity is very much equidistributed. The worst normalized difference occurs in region C at $n = 78823$, with 39816 odds and only 39007 evens. (On Table 7, this n lies between the first two entries, and has a ratio = 1.02074.)

It is reasonable to conjecture that

$$\text{(15)} \qquad \psi_1(n) = O(n^\epsilon)$$

for any positive ϵ. If this is true, then we have not merely that the ratio $\to 1$, but we also know its rate of convergence:

$$\text{(16)} \qquad |\text{ratio} - 1| < an^{-1/2+\epsilon}$$

for some a, and any ϵ.

7. Runs. The data in Section 5 was extended only to 8-tuples. To go beyond would require massive amounts of data, but the following special cases are of some interest. How often should one expect say, 15, *and only* 15 consecutive odd partition numbers? Since this presumes that the partition numbers immediately prior to such a sequence and immediately subsequent are both even, we are in fact asking for the count of 17-tuples of type $2(2^{15} - 1) = 65534$. As above, the expectation to $n = 2\cdot 10^6$ is

$$\sum_{65534}^{(17)} (2\cdot 10^6) = 2^{-17}(2\cdot 10^6) = 15.26\,.$$

Actually, there are 16 such runs of exactly 15 successive odds—the first run beginning with $p(108417)$, and the sixteenth beginning with $p(1936252)$.

In Table 9 we indicate the number of runs $\geqq 15$ out to $n = 2\cdot 10^6$. There are no runs here greater than 20. All of this data seems to be as expected.

TABLE 9

k	*Even Runs*	*Odd Runs*	*Expectation*
15	10	16	15.3
16	7	4	7.6
17	5	5	3.8
18	2	4	1.9
19	2	0	1.0
20	1	0	0.5
Total	27	29	30.1

Curio-collectors may wish to know that the 20 partition numbers

$$p(n)\,, \qquad 1517214 \leqq n \leqq 1517233$$

are all even, while

$$p(n)\,, \qquad 617995 \leqq n \leqq 618012$$

constitutes the first sequence of exactly 18 odd partition numbers.

8. Remarks on the Presumed Normality. The last three sections, taken together, do make a good empirical case for normality (modulo 2). We are indebted to Dr. A. O. L. Atkin for a reason why the modulus 2 and also the modulus 3 would be expected to be special for the partition numbers. All known congruence relations for these numbers can be deduced from the so-called *modular forms*. Entering here in a fundamental way is the linear function

$$24m - 1\,,$$

and while this can be divisible by any prime greater than 3, 2 and 3 are clearly special. Therefore, Atkin would also expect normality (modulo 3). We have not examined this.

Of course, such considerations are merely suggestive, and, so far, have not led to a *proof* of normality for either modulus, 2 or 3.

Another aspect of the distinction here between the apparent normality (modulo 2) and the distinct nonnormality (modulo 5), as exemplified in (2) and (5), is that one is reminded of the numbers of Wolfgang Schmidt. As is known, he showed [15], [16] that there exist real numbers x normal to one base r without being normal to another s. Perhaps we should clarify the difference between the phenomena presently under investigation and Schmidt's phenomena. Given any sequence of integers, $a(n)$, we could construct *two different real numbers* as in our equations (3) and (4), and they may, as apparently is the case here, be normal to one base while not to another. On the other hand, a Schmidt number x gives rise to *two different integer sequences:*

$$a(n) = [xr^n] \quad \text{and} \quad b(n) = [xs^n]\,.$$

Finally, we wish to draw the main inference. Some time ago, Professor Freeman Dyson wrote one of us, "Atkin and I were never able to do anything with modulo 2 [for the partition function]." But if the parity is normal, and this is what our investigation strongly suggests, it appears to be a valid inference that "nothing" can be done—"nothing" surely as simple as the congruence (2), or even as profound as the congruence (5). There remains the problem of *proving* the presumed normality, but no doubt that will be very difficult. Rather more promising is the weaker problem of showing that every k-tuple occurs, that is:

$$\sum_t^{(k)} (n) > 0 \qquad (\text{every } t,\, k)$$

for a sufficiently large n. Happily, this implies the (only seemingly stronger) result:

$$\sum_t^{(k)} (n) \to \infty \qquad (\text{all } t,\, k)\,.$$

9. Iterated Computation of the Parity; An Unsolved Problem. As we indicated at the end of Section 3, by iterating equations (11) one can determine individual parities independently of any stored table of a_n except for

$$a_0 = 1\,, \qquad a_2 = 0\,.$$

This leads to an unsolved problem of interest. Let us introduce an abbreviated notation; instead of

$$a_{200} \equiv a_{50} + a_{43} + a_{41} + a_{20} + a_{16}$$

we write

$$200 = 50, 43, 41, 20, 16\,.$$

The algorithm is standardized by use of the three rules:

(a) Replace the largest term on the right by its equivalent in (11).

(b) Whenever two repetitions of an argument appear on the right, cancel them both (since their sum is even in any case).

(c) Repeat until 0 or 2 or 0, 2 is all that remains on the right. Example:

For 200 one has the sequence:

50, *43*, *41*, *20*, *16*, *11*, **10**, **10**, *7*, *10*, **5**, **5**, *4*, 2, **1**, **0**, **1**, *1*, **0**.

Here we have italicized each term replaced by its equivalent, and used boldface for each pair eliminated by cancelling. Thus $p(200) \equiv p(2) =$ even.

In the computation for 200 we listed 19 *terms*, and *cancelled* 4 pairs. We define

$$t(n) \qquad \text{and} \qquad c(n)$$

to be these two functions. Thus

$$t(200) = 19\,, \qquad c(200) = 4\,.$$

Let us compute these functions for $n = 100, 200, 300, 400, 500, 600$. To do the algorithm efficiently, it is best not to use (11) directly, but, after having decided whether the current term to be replaced is of the form

$$4n\,, \quad 4n+1\,, \quad 4n+3\,, \quad \text{or} \quad 4n+6\,,$$

respectively, we write down n, and then subtract according to the differences:

7, 2, 21, 4, 35, 6, 49, 8, etc.,
5, 6, 15, 12, 25, 18, 35, 24, etc.,
3, 10, 9, 20, 15, 30, 21, 40, etc., or
1, 14, 3, 28, 5, 42, 7, 56, etc.

Here is a brief Table 10.

TABLE 10

n	$t(n)$	$c(n)$
100	11	2
200	19	4
300	30	9
400	38	11
500	58	16
600	56	17

We raise the questions whether

$$t(n) = O(n)? \tag{17}$$

$$c(n) = O(n)? \tag{18}$$

Clearly, $t(n)$ will generally increase with n, but "luck" plays a part; for 400 and 600 there is much cancellation of large terms, while for 500 there is relatively little.

The real point of our query is the question whether the parity of an individual $p(n)$ can be determined in $O(n)$ operations. If one computed such an *individual* parity by our previous, *en masse*, table building, technique the computation would require

$$\int O(\sqrt{n})dn = O(n^{3/2})$$

operations. We do not know whether (17) is true.

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